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Sets of Best L_1-Approximants

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1. INTRODUCTION

Let $X = L_1(\Omega, \mathscr{A}, \mu)$ and let $\mathscr{C} \subseteq X$ be an L_1-closed, convex subset. We say $g \in \mathscr{C}$ is a best L_1-approximant to $f \in X$ if $\|g - f\|_1 = \inf \|h - f\|_1$, $h \in \mathscr{C}$. For many important choices of $\mathscr{C}$, such as $\mathscr{C} = L_1(\Omega, \mathscr{B}, \mu)$, where $\mathscr{B}$ is a sub-σ-algebra of $\mathscr{A}$, or $\mathscr{C}$ the set of nondecreasing functions on $\Omega = [0, 1]$, best L_1-approximants exist to all $f \in X$. It is rare, however, that best L_1-approximants are uniquely determined. Denote by $\mu_1(f \mid \mathscr{C})$ the set of all best L_1-approximants to f by elements of $\mathscr{C}$. In this paper we study the question: If f_1 and f_2 are "close," are the sets $\mu_1(f_1 \mid \mathscr{C})$ and $\mu_1(f_2 \mid \mathscr{C})$ "close" in Hausdorff metric?

2. APPROXIMATION BY ELEMENTS OF $L_1(\Omega, \mathscr{B}, \mu)$

Let $\mathscr{B}$ be a sub-σ-algebra of $\mathscr{A}$, and let $\mathscr{C} = L_1(\Omega, \mathscr{B}, \mu)$. Shintani and Ando [4, Theorem 2] proved the existence of best L_1-approximations to $f \in X = L_1(\Omega, \mathscr{A}, \mu)$ by elements of $\mathscr{C}$. Furthermore, they characterized the set $\mu_1(f \mid \mathscr{C})$ in the following way: there exist functions $\bar{f}$ and $\underline{f}$ in $\mathscr{C}$ such that $g \in \mu_1(f \mid \mathscr{C})$ if and only if $g \in \mathscr{C}$ and $\underline{f} \leqslant g \leqslant \bar{f}$ on Ω. In particular, $\bar{f} = \sup\{g : g \in \mu_1(f \mid \mathscr{C})\}$ and $\underline{f} = \inf\{g : g \in \mu_1(f \mid \mathscr{C})\}$.

If A is a subset of a metric space M with distance d, define $\operatorname{dist}(x, A) = \inf\{d(x, a) : a \in A\}$. If A and B are subsets of M, define the Hausdorff distance between them by $\operatorname{dist}(A, B) = \max\{\sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A)\}$.

The most natural question at this point is: If $f_n \to f$ in L_1 as $n \to \infty$, does $\operatorname{dist}(\mu_1(f_n \mid \mathscr{C}), \mu_1(f \mid \mathscr{C})) \to 0$ as $n \to \infty$, where $d(g, h) = \|g - h\|_1$? The following example shows in general the answer is no.

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EXAMPLE 2.1. Let $\Omega = [0, 1]$ with Lebesgue measure and $\mathscr{B} = \{\phi, \Omega\}$. Then g is $\mathscr{B}$-measurable if and only if g is constant. Define $f(x)$ by $f(x) = 1$ on $[0, \frac{1}{2})$ and $f(x) = 0$ on $[\frac{1}{2}, 1]$. For $n \geqslant 3$, define $f_n(x)$ by $f_n(x) = 1$ on $[0, \frac{1}{2} + 1/n)$ and $f_n(x) = 0$ on $[\frac{1}{2} + 1/n, 1]$. Then clearly $f_n \to f$ in L_1, and each f_n has a unique best L_1-approximant defined by $g_n(x) = 1$ on $[0, 1]$. But $f(x)$ has many best L_1-approximants, defined by $g_c(x) = c$ on $[0, 1]$, where $0 \leqslant c \leqslant 1$. In particular, $g_0 \in \mu_1(f \mid \mathscr{C})$ and $\operatorname{dist}(g_0, \mu_1(f_n \mid \mathscr{C})) = 1$ for all $n \geqslant 3$. Hence $\operatorname{dist}(\mu_1(f \mid \mathscr{C}), \mu_1(f_n \mid \mathscr{C})) \geqslant 1$ for all $n \geqslant 3$. (Clearly this is an equality.)

We can, however, prove the following semi-continuity result.

THEOREM 2.2. *Let $f_n \to f$ in L_1 as $n \to \infty$ and let $\varepsilon > 0$. There is an $N > 0$ such that* $\operatorname{dist}(g, \mu_1(f \mid \mathscr{C})) < \varepsilon$ *for all $g \in \mu_1(f_n \mid \mathscr{C})$ with $n \geqslant N$.*

Proof. By Shintani and Ando [4, Corollary 5], we have $\bar{f}_n \vee \bar{f} \to \bar{f}$ in L_1 as $n \to \infty$ and $\underline{f}_n \wedge \underline{f} \to \underline{f}$ in L_1 as $n \to \infty$. Choose N such that $\|\bar{f}_n \vee \bar{f} - \bar{f}\|_1 < \varepsilon/2$ and $\|\underline{f}_n \wedge \underline{f} - \underline{f}\|_1 < \varepsilon/2$ for $n \geqslant N$. Now if $n \geqslant N$ and $g \in \mu_1(f_n \mid \mathscr{C})$, define $g^* = \underline{f} \vee g \wedge \bar{f}$. Then $g^* \in \mu_1(f \mid \mathscr{C})$. Since $\underline{f}_n \leqslant g \leqslant \bar{f}_n$, it follows that $g^* = g$ except possibly on the sets $A = \{\underline{f}_n < \underline{f}\}$ and $B = \{\bar{f}_n > \bar{f}\}$. Hence

$$\begin{aligned}\|g^* - g\|_1 &= \int_{A \cup B} |g^* - g| \, d\mu \leqslant \int_A |\underline{f}_n - \underline{f}| \, d\mu + \int_B |\bar{f}_n - \bar{f}| \, d\mu \\ &= \int_A |\underline{f}_n \wedge \underline{f} - \underline{f}| \, d\mu + \int_B |\bar{f}_n \vee \bar{f} - \bar{f}| \, d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

and the theorem is proved.

If we use the uniform metric defined by $d(g, h) = \|g - h\|_\infty$, we may obtain the full continuity result.

THEOREM 2.3. *Let $f_n \to f$ uniformly as $n \to \infty$. Then* $\operatorname{dist}(\mu_1(f_n \mid \mathscr{C}), \mu_1(f \mid \mathscr{C})) \to 0$ *as $n \to \infty$.*

Proof. By Landers and Rogge [3, Theorem 18] the mappings $f \to \bar{f}$ and $f \to \underline{f}$ are monotone, which implies $\bar{f}_n \to \bar{f}$ and $\underline{f}_n \to \underline{f}$ uniformly as $n \to \infty$. If $\varepsilon > 0$, choose N such that $|\bar{f}_n - \bar{f}| < \varepsilon$ and $|\underline{f}_n - \underline{f}| < \varepsilon$ on Ω for $n \geqslant N$. Then if $g \in \mu_1(f \mid \mathscr{C})$, define $g^* = \underline{f} \vee g \wedge \bar{f}$. Then $g^* \in \mu_1(f_n \mid \mathscr{C})$ and $|g^* - g| < \varepsilon$ on Ω for $n \geqslant N$. If $g \in \mu_1(f_n \mid \mathscr{C})$, define $g^* = \underline{f} \vee g \wedge \bar{f}$. Then $g^* \in \mu_1(f \mid \mathscr{C})$ and $|g^* - g| < \varepsilon$ on Ω for $n \geqslant N$. Hence $\operatorname{dist}(\mu_1(f_n \mid \mathscr{C}), \mu_1(f \mid \mathscr{C})) < \varepsilon$ for $n \geqslant N$.

3. Approximation by Nondecreasing Functions

Let $\mathcal{N}$ be the set of nondecreasing functions on $[0, 1]$, and suppose $f \in L_1[0, 1]$. The set $\mu_1(f \mid \mathcal{N})$ of all best L_1-approximations to f by elements of $\mathcal{N}$ is characterized in [1, 2] as follows. Define $\bar{f}$ and $\underline{f}$ by $\bar{f}(x) = \sup\{q(x): q \in \mu_1(f \mid \mathcal{N})\}$ and $\underline{f}(x) = \inf\{q(x): q \in \mu_1(f \mid \mathcal{N})\}$. It is shown in [3, Theorem 14] that $\bar{f}$ and $\underline{f}$ are in $\mu_1(f \mid \mathcal{N})$. Let $U = \bigcup U_i$, where U_i is a maximal open interval on which both $\bar{f}$ and $\underline{f}$ are constant and $\bar{f} \neq \underline{f}$. Define $h_f: U \to R$ by

$$h_f(x) = \begin{cases} 1 & \text{if } f(x) \geqslant \bar{f}(x) \\ -1 & \text{if } f(x) \leqslant \underline{f}(x) \\ 0 & \text{if } \underline{f}(x) < f(x) < \bar{f}(x), \end{cases}$$

and, if $x \in U_i = (u_i, v_i)$, define k_f by

$$k_f(x) = \int_{u_i}^{x} h_f(t)\, dt.$$

Then for any $q \in \mathcal{N}$, we have $q \in \mu_1(f \mid \mathcal{N})$ if and only if

(i) $\underline{f} \leqslant g \leqslant \bar{f}$ on $[0, 1]$, and

(ii) q is constant on components of $\{[k_f \neq 0] \cap U_i\}$, $i \geqslant 1$.

We use the notation $\mu(A; [a, b])$ to denote $\mu(A)/(b-a)$, the relative measure of A in $[a, b]$. The following lemma was proved in [2] and will be used later in this paper.

Lemma 3.1. *If $q \in \mu_1(f \mid \mathcal{N})$ and q is not constant at $s \in [0, 1]$, then*

(1) $\mu([f \geqslant q]; [s, t]) \geqslant \frac{1}{2}$ *for* $s < t \leqslant 1$, *and*

(2) $\mu([f \leqslant q]; [t, s]) \geqslant \frac{1}{2}$ *for* $0 \leqslant t < s$.

The main result of this section is an easy consequence of the following lemma.

Lemma 3.2. *Suppose $\varepsilon > 0$ and $f, g \in L_1[0, 1]$. If $|f(x) - g(x)| < \varepsilon$ for all $0 \leqslant x \leqslant 1$, then for any $f^* \in \mu_1(f \mid \mathcal{N})$ there is a $g^* \in \mu_1(g \mid \mathcal{N})$ so that $|f^*(x) - g^*(x)| < 8\varepsilon$ for all $0 \leqslant x \leqslant 1$.*

Proof. We have by [3, Theorem 18] that $|\bar{g}(x) - \bar{f}(x)| < \varepsilon$ and $|\underline{g}(x) - \underline{f}(x)| < \varepsilon$ for all $0 \leqslant x \leqslant 1$.

Let U and U_i, for $i \geqslant 1$, be defined as above for f, and let V and V_i, for $i \geqslant 1$, be defined as above for g. Let $U(\varepsilon) = \bigcup_{i \in I(\varepsilon)} U_i$, where $I(\varepsilon)$ is the set of indices such that $\bar{f}(x) - \underline{f}(x) > 6\varepsilon$ for $x \in U_i$. Since $\bar{f}$ is continuous from the right and $\underline{f}$ is continuous from the left, it follows that $\bar{f}(x)-\underline{f}(x)>6\varepsilon$ for $x\in \bar{U}_i$. For $f^*\in\mu_1(f\,|\,\mathcal{N})$ we define g^* as follows: if $x\in U(\varepsilon)$, then $g^*(x)=\underline{g}(x)\vee f^*(x)\wedge \bar{g}(x)$; if $x<y$ for all $y\in U(\varepsilon)$, then $g^*(x)=\underline{g}(x)$; and otherwise,

$$g^*(x)=\Big(\sup_{\substack{y<x\\ y\in U(\varepsilon)}} g^*(y)\Big)\vee \underline{g}(x).$$

It is clear from the definition of g^* that $\underline{g}(x)\leqslant g^*(x)\leqslant \bar{g}(x)$ for all $0\leqslant x\leqslant 1$. Thus $g^*(x)$ will be in $\mu_1(g\,|\,\mathcal{N})$ provided

$$g^* \text{ is constant on components of } \{V_i\cap[k_g\neq 0]\},\ i\geqslant 1. \tag{1}$$

Suppose (1) is not true. Then there is an $x_0\in V_j$ for some j so that $k_g(x_0)\neq 0$ and g^* is not constant at x_0. Since g^* is constant on maximal components of the complement of $\overline{U(\varepsilon)}$, where $\overline{U(\varepsilon)}$ is equal to either $U(\varepsilon)$ or $U(\varepsilon)\cup\{1\}$, we have $x_0\in U(\varepsilon)$ and, from the definition of g^*, f^* is not constant at x_0. It follows from Corollary 2 and Theorem 5 of [2] that either f^* has a jump discontinuity at x_0 or $\bar{f}(x)=\underline{f}(x)=f(x)$ almost everywhere in an interval containing x_0. Since $\bar{f}(x)\neq\underline{f}(x)$ for all $x\in U(\varepsilon)$, we have that f^*, and hence g^*, has a jump discontinuity at x_0. Clearly since $\bar{f}(x_0)-\underline{f}(x_0)>6\varepsilon$, we have that $\bar{g}(x_0)-\underline{g}(x_0)>4\varepsilon$ and hence, $\bar{g}(x)-\underline{g}(x)>4\varepsilon$ for all $x\in V_j$. It follows that $\bar{f}(x)-\underline{f}(x)>2\varepsilon$ for all $x\in V_j$, and thus $\mu(V_j-U)=0$. Also, it is shown in [2] that

$$\mu([\underline{f}<f<\bar{f}]\cap V_j)=0 \qquad\text{and}\qquad \mu([\underline{g}<g<\bar{g}]\cap V_j)=0. \tag{2}$$

We now show that for almost all $x\in V_j\cap U$ we have

$$h_g(x)=h_f(x). \tag{3}$$

If $h_g(x)=1$, then $f(x)>g(x)-\varepsilon\geqslant\bar{g}(x)-\varepsilon\geqslant\bar{f}(x)-2\varepsilon>\underline{f}(x)$. In view of (2) we have $f(x)\geqslant\bar{f}(x)$ for almost all such x, implying $h_f(x)=1$. On the other hand, if $h_f(x)=1$, then $g(x)>f(x)-\varepsilon\geqslant\bar{f}(x)-\varepsilon>\bar{g}(x)-2\varepsilon>\underline{g}(x)$. In view of (2) we have $g(x)\geqslant\bar{g}(x)$ for almost all such x, implying $h_g(x)=1$. The proof that $h_f(x)=-1$ if and only if $h_g(x)=-1$ for almost all x for which $h_f(x)=-1$ or $h_g(x)=-1$ is similar, and (3) follows.

Now if $V_j=(w,z)$ then $G=\frac{1}{2}(\bar{g}+\underline{g})$ is not constant at w. From Lemma 3.1 we have $\mu([g\geqslant G];[w,x_0])\geqslant\frac{1}{2}$, and in view of (2), $\mu([g\geqslant\bar{g}];[w,x_0])\geqslant\frac{1}{2}$, implying

$$\int_w^{x_0} h_g(t)\,dt\geqslant 0. \tag{4}$$

Also since f^* is not constant at x_0, we have from Lemma 3.1 that $\mu([f \leqslant f^*]; [w, x_0]) \geqslant \frac{1}{2}$. In view of (2), $\mu([f \leqslant \underline{f}]; [w, x_0]) \geqslant \frac{1}{2}$, implying $\int_{[w,x_0] \cap U} h_f(t)\, dt \leqslant 0$. It follows from (3) and the fact that $(V_j - U) = 0$ that

$$\int_{w}^{x_0} h_g(t)\, dt \leqslant 0. \tag{5}$$

From (4) and (5) we see that $\int_w^{x_0} h_g(t)\, dt = 0$, implying that $k_g(x_0) = 0$, a contradiction. Thus (1) is proved and $g^* \in \mu_1(g \mid \mathcal{N})$.

We now show that $|g^* - f^*| < 8\varepsilon$ for all $x \in [0, 1]$. We have that $|\bar{f}(x) - \bar{g}(x)| < \varepsilon$ and $|\underline{f}(x) - \underline{g}(x)| < \varepsilon$ for all $x \in [0, 1]$. If $x \in U(\varepsilon)$, then $g^*(x)$ equals $f^*(x)$, $\bar{g}(x)$, or $\underline{g}(x)$. If $g^*(x) = \bar{g}(x)$ then from the definition of g^* we have $\bar{g}(x) \leqslant f^*(x) \leqslant \bar{f}(x)$. Thus $|g^*(x) - f^*(x)| = |\bar{g}(x) - f^*(x)| \leqslant |\bar{g}(x) - \bar{f}(x)| < \varepsilon$. If $g^*(x) = \underline{g}(x)$, then again from the definition of g^* we have $\underline{f}(x) \leqslant f^*(x) \leqslant \underline{g}(x)$. Thus $|g^*(x) - f^*(x)| = |\underline{g}(x) - f^*(x)| \leqslant |\underline{g}(x) - \underline{f}(x)| < \varepsilon$. On the other hand, if $x \in U(\varepsilon)$, then $|\bar{f}(x) - \underline{f}(x)| \leqslant 6\varepsilon$. Thus $|f^*(x) - g^*(x)| \leqslant \max(\bar{f}(x), \bar{g}(x)) - \min(\underline{f}(x), \underline{g}(x)) \leqslant 8\varepsilon$, and the lemma is proved.

The following theorem is an easy consequence of Lemma 3.2.

THEOREM 3.3. *For any* $\varepsilon > 0$, *if* $f, g \in L_1$ $[0, 1]$ *satisfy* $|f(x) - g(x)| < \varepsilon/8$ *for all* $0 \leqslant x \leqslant 1$, *then* $\operatorname{dist}(\mu_1(f \mid \mathcal{N}), \mu_1(g \mid \mathcal{N})) < \varepsilon$ *in the uniform metric.*

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